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1972 J. Phys. A: Gen. Phys. 5 1233

(http://iopscience.iop.org/0022-3689/5/8/015)

J. Phys. A: Gen. Phys., Vol. 5, August 1972. Printed in Great Britain.

S-wave eigenvalues of the Lippmann–Schwinger kernel for the exponential potential

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MS received 2 March 1972

Abstract. It is shown that the S-wave eigenvalues of the Lippmann–Schwinger kernel for the exponential potential can be determined from the zeros of a Bessel function whose order is energy dependent. An asymptotic expansion is obtained for the leading eigenvalue at negative energies.

1. Introduction

A knowledge of the eigenvalues $\eta(k^2)$ of the Lippmann–Schwinger kernel is usually required when the two-body off-shell partial-wave T matrix is evaluated from an eigenfunction expansion (k^2 is the energy in the centre of mass frame). Analytical and numerical results for the $\eta(k^2)$ of Coulomb, Hulthén, Yukawa, rectangular well, and Yamaguchi potentials have been presented by various authors (Weinberg 1963, Wright and Scadron 1964, Warburton 1966, Bierter and Dietrich 1967a, 1967b, Warburton and Stern 1969, Stern 1969, Fuda 1969); a numerical method which is suitable for application to any analytic local interaction has also been developed (Stern and Warburton 1972).

2. The exponential potential

In this section we shall obtain analytical results for the S-wave eigenvalues of the scattering kernel for the exponential potential

$$V(r) = -A \exp(-\mu r). \tag{1}$$

In this case the $\eta(k^2)$ are determined from the partial-wave Schrödinger equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \left(k^2 + \frac{A \exp(-\mu r)}{\eta(k^2)}\right)u = 0 \tag{2}$$

which is subject to the boundary conditions

$$\begin{aligned} &u(r) \sim r \qquad \text{as } r \to 0 \\ \text{and} \qquad &u(r) \sim \exp(\mathrm{i}kr) \qquad \text{as } r \to \infty \end{aligned} \tag{3}$$

where we take $\operatorname{Im} k \geqslant 0$, corresponding to the k^2 plane cut along the positive real axis.

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The method we shall employ to solve equation (2) is similar to that used by Wu and Ohmura (1962, p 94) for the case $\eta(k^2) = 1$. The substitutions

$$\nu = \frac{2ik}{\mu} \qquad \alpha = \frac{2}{\mu}\left(\frac{A}{\eta}\right)^{1/2} \qquad y = \alpha \exp\left(-\frac{\mu r}{2}\right) \tag{4}$$

transform the differential equation to

$$y^2\frac{d^2u}{dy^2} + y\frac{du}{dy} + (y^2 - \nu^2)u = 0 \tag{5}$$

which is Bessel's equation of order ν. The solution is

$$u(y) = -\frac{C}{J_\nu(\alpha)}(J_{-\nu}(\alpha)J_\nu(y) - J_\nu(\alpha)J_{-\nu}(y)) \tag{6}$$

since $y = \alpha$ at $r = 0$. C is a constant. From the well-known power series for $J_{\pm\nu}(y)$ we find that as $r \to \infty$ (ie $y \to 0$) the wavefunction takes the asymptotic form

$$u(r) \sim -\frac{C}{J_\nu(\alpha)}\left\{\left(\frac{\alpha}{2}\right)^{\nu}\frac{J_{-\nu}(\alpha)}{\Gamma(\nu+1)}\exp(-ikr) - \left(\frac{\alpha}{2}\right)^{-\nu}\frac{J_\nu(\alpha)}{\Gamma(-\nu+1)}\exp(ikr)\right\}. \tag{7}$$

However, from the second of the boundary conditions (3) we see that the term containing $\exp(-ikr)$ should be absent, which implies that

$$J_{-\nu}(\alpha) = 0 \tag{8}$$

where α and ν are defined in (4). Thus the eigenvalues $\eta(k^2)$ are obtained from the zeros of $J_{-\nu}(\alpha)$, a Bessel function of energy-dependent order. The wavefunction (6) therefore reduces to

$$u(r) = CJ_{-\nu}(y). \tag{9}$$

With the aid of the well known relation

$$J_{1/2}(\alpha) = \left(\frac{2}{\pi\alpha}\right)^{1/2}\sin\alpha \tag{10}$$

it is found that at $k = i\mu/4$ ($\nu = -\frac{1}{2}$) equation (8) yields the result

$$\eta\left(-\frac{\mu^2}{16}\right) = \frac{4A}{N^2\pi^2\mu^2} \qquad (N = 1, 2, 3, \ldots). \tag{11}$$

N is the order of eigenvalue and we can thus speak of the Nth eigenvalue $\eta_N(k^2)$. This formula shows that the S-wave eigenvalues of the exponential potential decrease as N^{-2} as do those of Yukawa and Hulthén potentials (Warburton and Stern 1969).

When $A > 0$ the potential (1) will form a physical bound state if $\eta_N(k^2) = 1$ for a value of k on the positive imaginary axis in the complex k plane (Weinberg 1963). There is one bound state for each eigenvalue which is not less than 1 at zero energy. Thus, by setting η equal to unity in equations (4) and (8), it is seen that S-wave binding energies can be found from the roots of the equation

$$J_{-\nu}\left(\frac{2A^{1/2}}{\mu}\right) = 0 \tag{12}$$

which has also been derived via another method by Massey and Mohr (1935).

We shall now determine the high-energy behaviour of the leading eigenvalue $\eta_1(k^2)$. An asymptotic expansion for the first positive zero of $J_{-\nu}(\alpha)$ for large positive $-\nu$—that is, for large values of k on the positive imaginary axis in the k plane—is

$$\frac{2}{\mu}\left(\frac{A}{\eta_1}\right)^{1/2} = -\nu\left(1+\sum_{n=1}^{\infty} c_n(-\nu)^{-2n/3}\right) \tag{13}$$

where the coefficients c_n have been given by Abramowitz and Stegun (1965). Thus we obtain the asymptotic expansion

$$\eta_1(k^2) = -\frac{A}{k^2}\sum_{m=0}^{\infty}(-1)^m(m+1)\left\{\sum_{n=1}^{\infty} c_n\left(-\frac{2ik}{\mu}\right)^{-2n/3}\right\}^m \tag{14}$$

for high negative energies, from which we note the high-energy limit $\eta_1(k^2) \sim 1/k^2$ in contrast to the Coulomb high-energy limit $\eta_N(k^2) \sim 1/k$ which is observed by the eigenvalues of local interactions with a $1/r$ singularity at the origin (Warburton and Stern 1969). At high energies the $\eta_N(k^2)$ of this latter class of potentials are represented by expansions in powers of $1/k$, while equation (14) shows that the leading eigenvalue of the exponential potential (1) is represented by a series in powers of $1/k^{2/3}$.

3. Conclusion

We have shown above that it is possible to obtain analytical representations of the S-wave eigenvalues of the Lippmann–Schwinger kernel for the exponential potential. However, for higher partial waves, we have only been able to compute the $\eta(k^2)$ of this interaction numerically (Stern and Warburton 1972). A knowledge of the eigenvalues $\eta(k^2)$ and corresponding wavefunctions $u(r)$ enables the two-particle off-shell partial-wave amplitude to be evaluated from an eigenfunction expansion separable in the off-shell momenta (Weinberg 1963, Warburton and Stern 1969). This is well known to be the most suitable form of the two-body T matrix to use in the three-particle Faddeev equations (Lovelace 1964, Bierter and Dietrich 1967a, 1967b, Fuda 1969, Karchenko *et al* 1970).

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